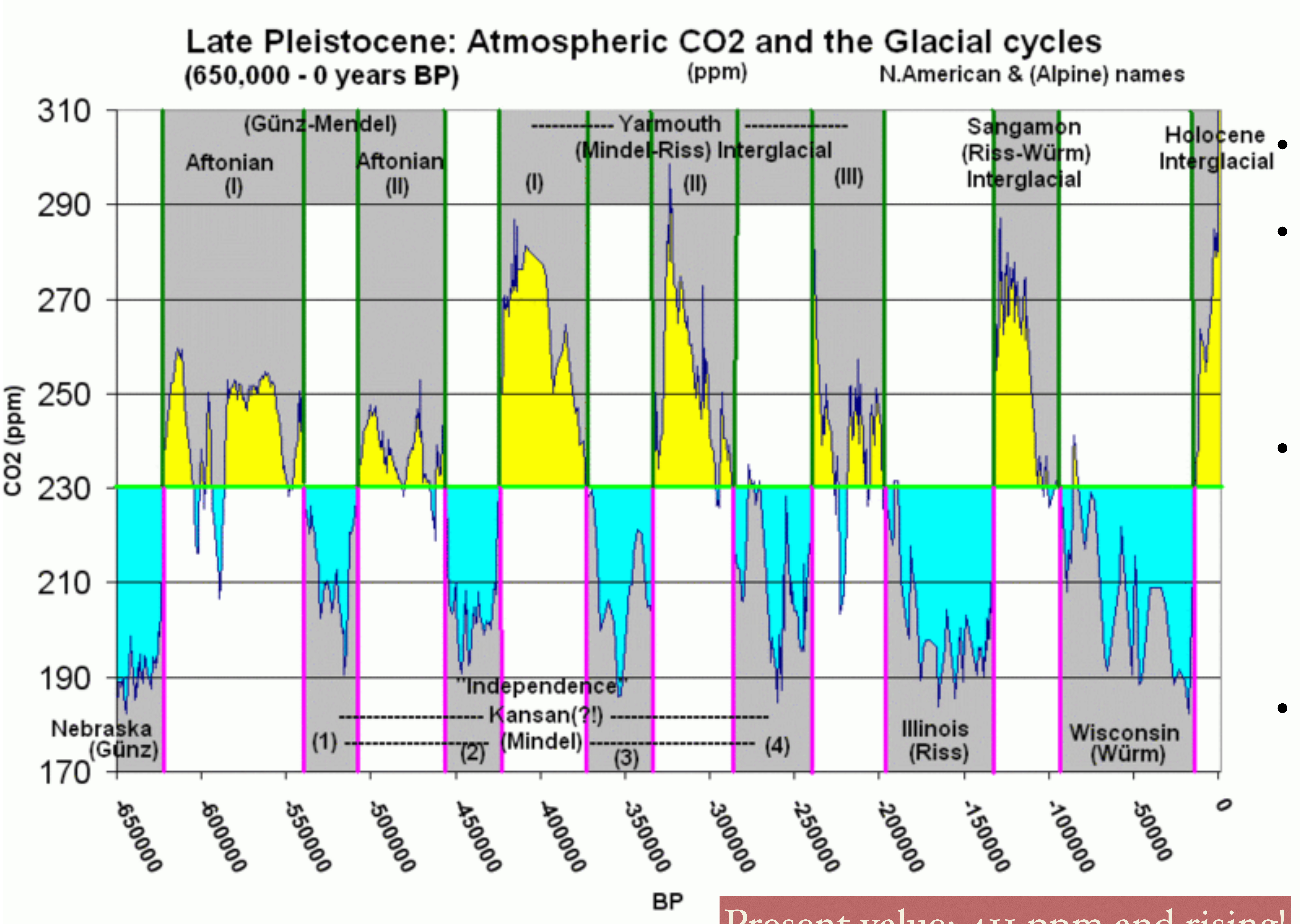
The Mathematics of Climate Change

Introduction

Marek Stastna

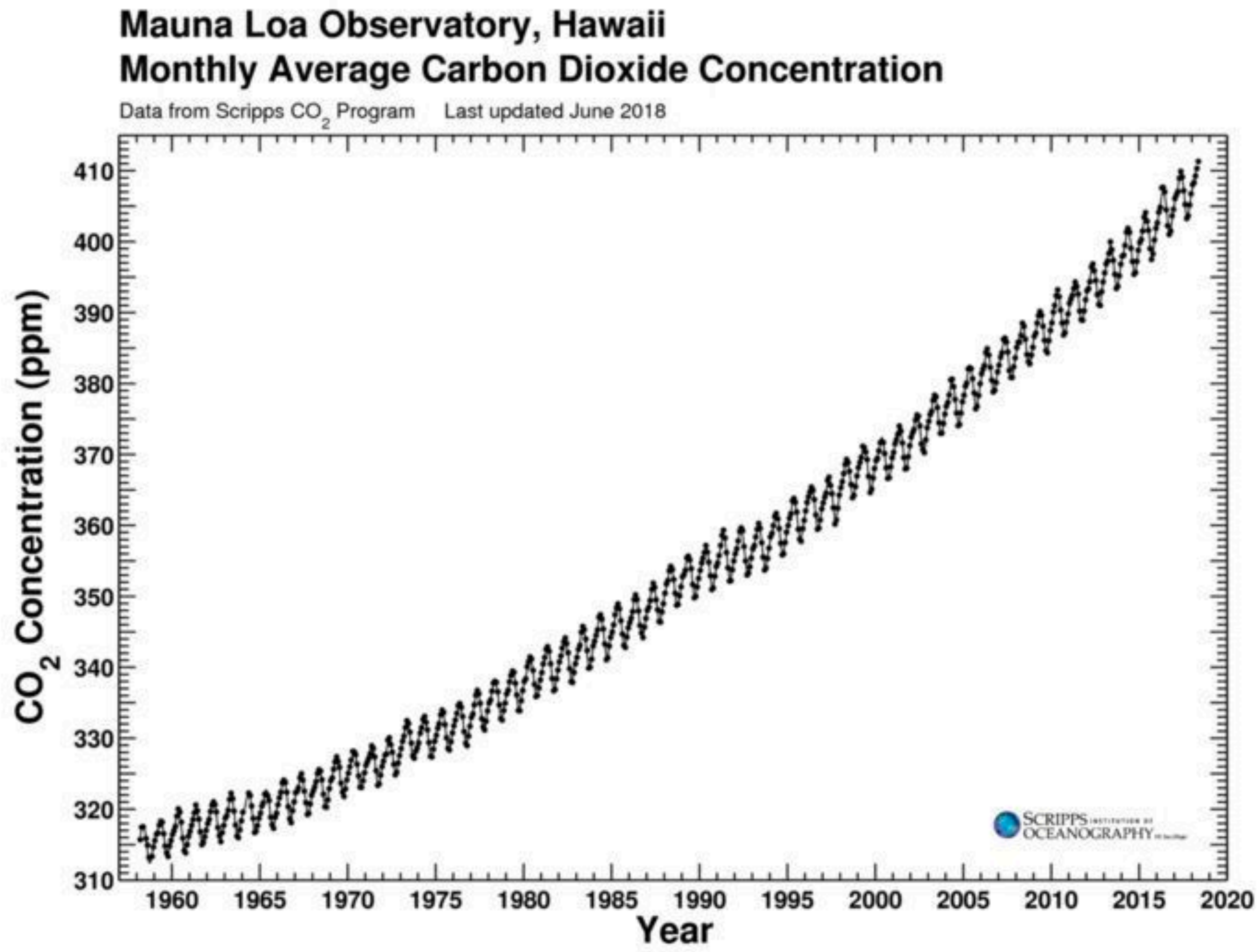


Present value: 411 ppm and rising!

- Climate changes on many timescales.
- We can get evidence about the past from geology, sedimentology and other sciences and get an idea of possible climates.
- We can also set up climate models of various sophistication to tune them to past climates and then try to predict the future.
- Mathematics is a big part of climate modelling but is often in the background

Recent CO_2

- Measurements since the 1960s show that the current atmospheric CO_2 concentration is increasing at an unprecedented rate.
- The actual values started at the very high end of those found over the last million years and have only gone up.
- Climate change is thus an O(1) effect (the clearest, most important part of the signal).



C. D. Keeling, S. C. Piper, R. B. Bacastow, M. Wahlen, T. P. Whorf, M. Heimann, and H. A. Meijer, Exchanges of atmospheric CO2 and 13CO2 with the terrestrial biosphere and oceans from 1978 to 2000. I. Global aspects, SIO Reference Series, No. 01-06, Scripps Institution of Oceanography, San Diego, 88 pages, 2001. http://escholarship.org/uc/item/09v319r9

There is a deeply human side to climate science that won't go away. And human caused climate change is NOT new:

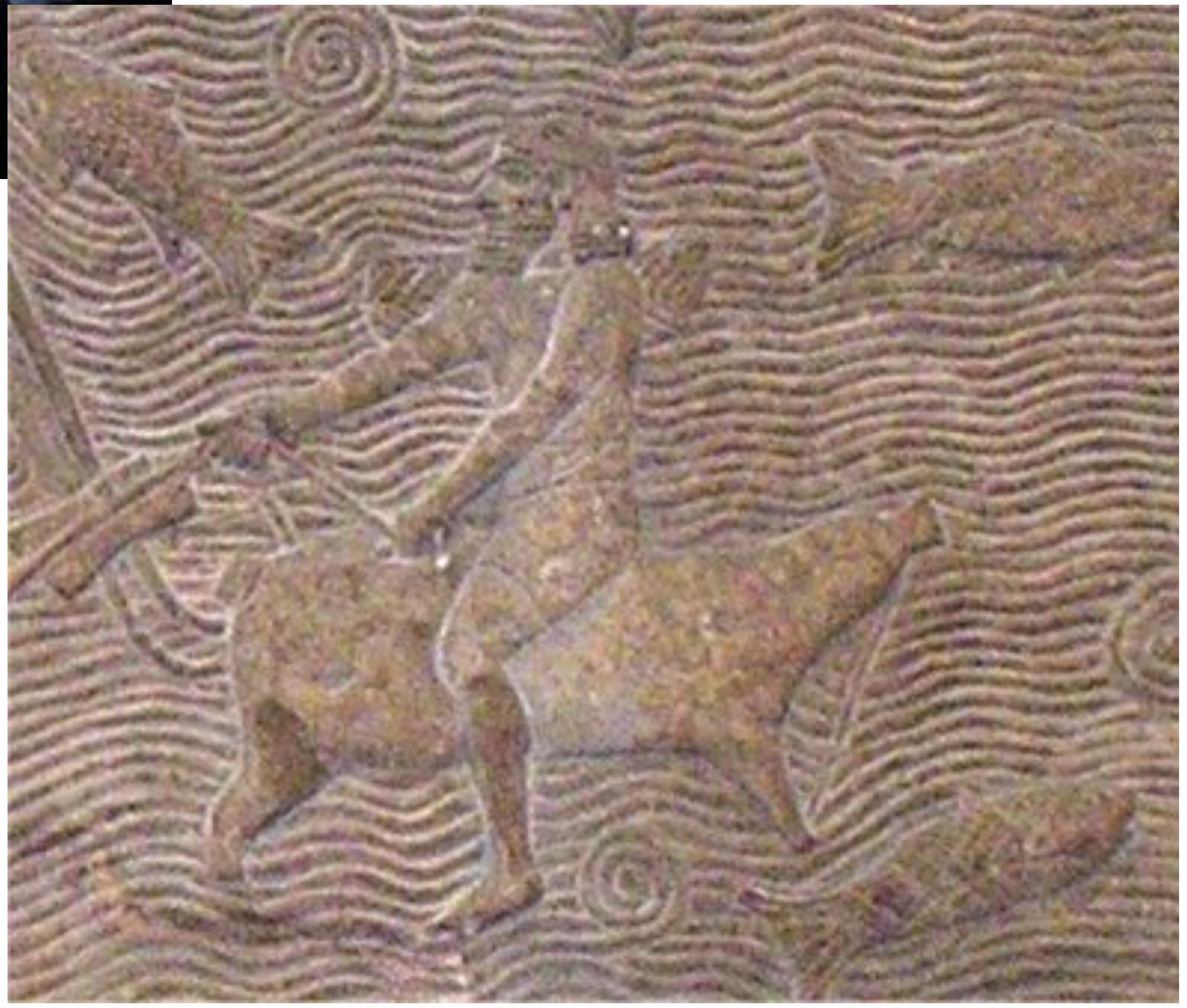
Aral Sea circa 1980

Aral Sea circa 2001

The Aral Sea

By I, Baburkhan, CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=2424235

The Amu Darya (Oxus) in recent times



Historical Account of Alexander the Great crossing the Oxus (Amu Darya): Bessus had tried to prevent the crossing of the Oxus by burning all available ships. However, the Macedonians made rafts. They stuffed animal skins and tents with hay, and five days later, the army was on the other bank in the southeast of what is now called Turkmenistan.

Climate’s many faces:

- Climate is a nonlinear physics problem: climate is an emergent, large number of degrees of freedom problem involving multiple branches of physics (radiative transfer, thermodynamics, fluid dynamics).
- Climate is a computer science problem since most of what we know about the “real” climate comes from climate models, which are computational in nature.
- Climate is a mathematics problem since so much of physics is intertwined with mathematics, and climate needs new types of vocabulary to describe it.
- Climate is a socio-economic problem since we experience climate and rely on some of its aspects to base our civilization on.
- Climate change is a very important aspect of climate, but only an aspect.

Climate’s challenges:

- Perhaps the most important aspect of climate science in terms of how it affects teaching the topic is that it has been heavily politicized.
- This politicization has connotations in how climate is taught (usually without much mathematics), and in how it is approached (as a controversy as opposed to a mix of challenges/opportunities).
- Climate change is an exponential process, meaning now that we are seeing its manifestations (e.g. wildfires in summer, atmospheric river shifts in winter) we are past the point where actionable items are “easy”.
- Climate is an emergent, nonlinear physical phenomenon and we still teach mathematics using conceptualizations that pre-date computers by a century or more.
- Climate change is also a justice issue and justice has a formal framework (law) that is different from mathematics.

This set of lectures will:

1. Introduce the understanding of climate change using slides and some YouTube movies.

2. Outline the functions and ideas that underpin the mathematical description of climate.

3. Survey one basic model of climate as a whole.

This set of lectures will not:

1. Replace general science, non-mathematical introductions to the climate system (because these provide the “facts” as opposed to the mathematical simplifications).
2. Make you a climate modelling expert.
3. Use a full, state of the art climate model.
4. Apologize for sticking to mathematical/computer science topics.

Climate Basics: Not the Weather

Weather

- Weather is without a doubt the natural process that has the most impact on our lives.
- Sometimes this is immediate, like forgetting an umbrella on a rainy day.
- Sometimes it is indirect, like when a drought in Brazil leads to higher coffee prices some time after the drought.
- Extreme weather can lead to direct harm (tornadoes, flash floods, etc).
- Some of the nastier effects of weather can be ameliorated by having strong societal safe guards in place, but the corollary is that it is often the poorest and most vulnerable that bear the brunt of weather events.
- The cumulative and long time trend of weather is what we call climate.

Weather: links

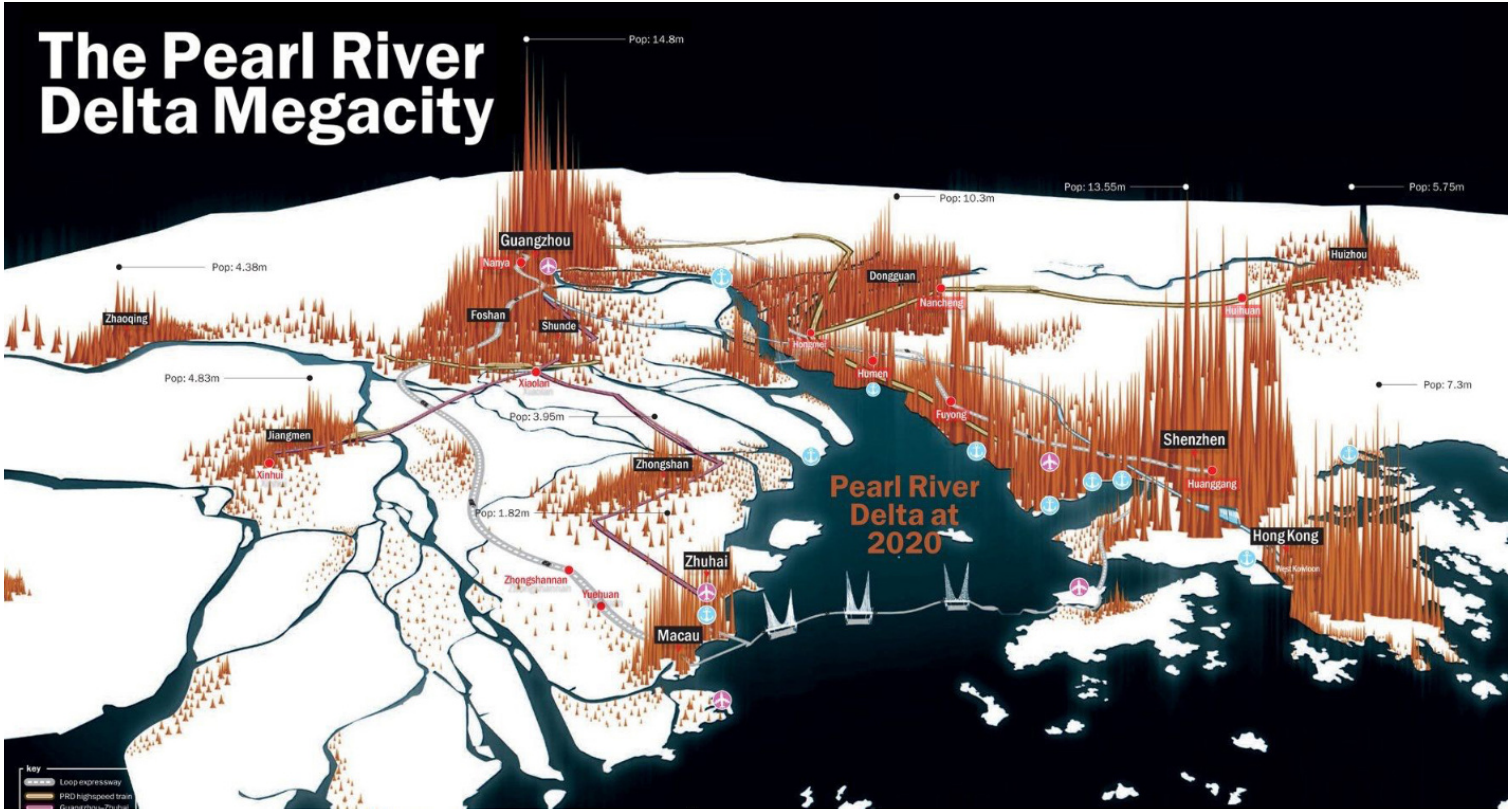
- The UW weather station https://weather.uwaterloo.ca/
- High resolution simulation: https://www.youtube.com/watch?v=4794mgJLTbU
- Environment Canada weather including forecatss, radar and satellite pictures: https://weather.gc.ca/canada_e.html

Weather: some final thoughts

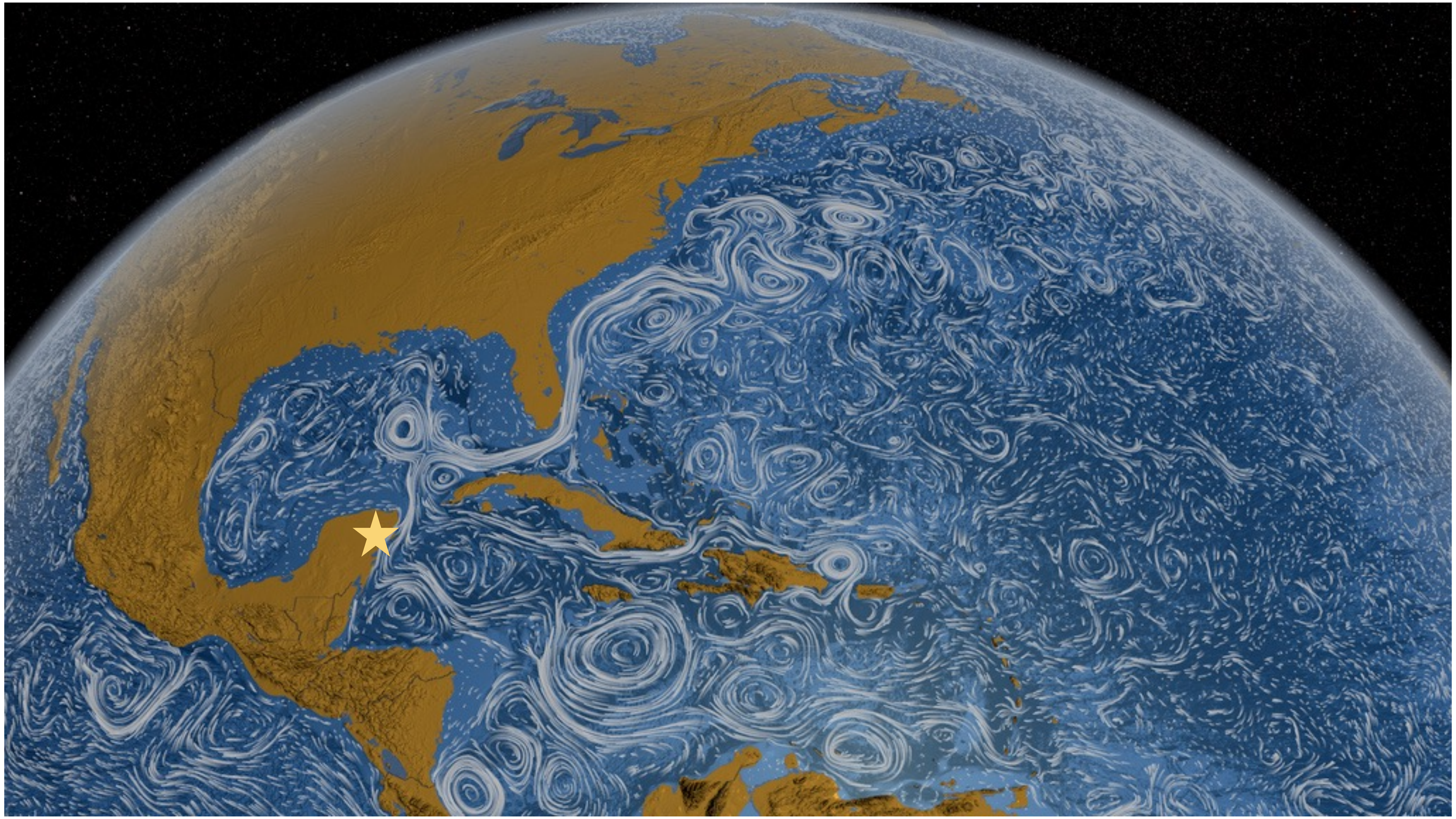
- Weather is super interesting, both as an almost constant topic of conversation, but also as a modern scientific/computing discipline.
- **It is part of a large economic sector both in the public and private sectors.**
- Some weather prediction involves amongst the largest simulations going.
- Other aspects involve statistics and other branches of mathematics.
- We won’t talk much about weather because in its modern form it is a branch of fluid dynamics, and as such there is a long pre-requisite chain of courses.
- I will happily discuss weather as a career with interested students, and share my social network (I was President of the Meteorological and Oceanographic Society).

- For some balance, and to give some sense of where populations affected by climate change I am going to show some pictures from space.
- Lake Erie is the shallowest, and most biologically productive, but Lake Ontario has similar algal blooms, both have large population centres and are effected by climate change.

- The Great Lakes basin is full of people, but around the world the near shore region is even more crowded as this picture of the Pearl River delta shows.



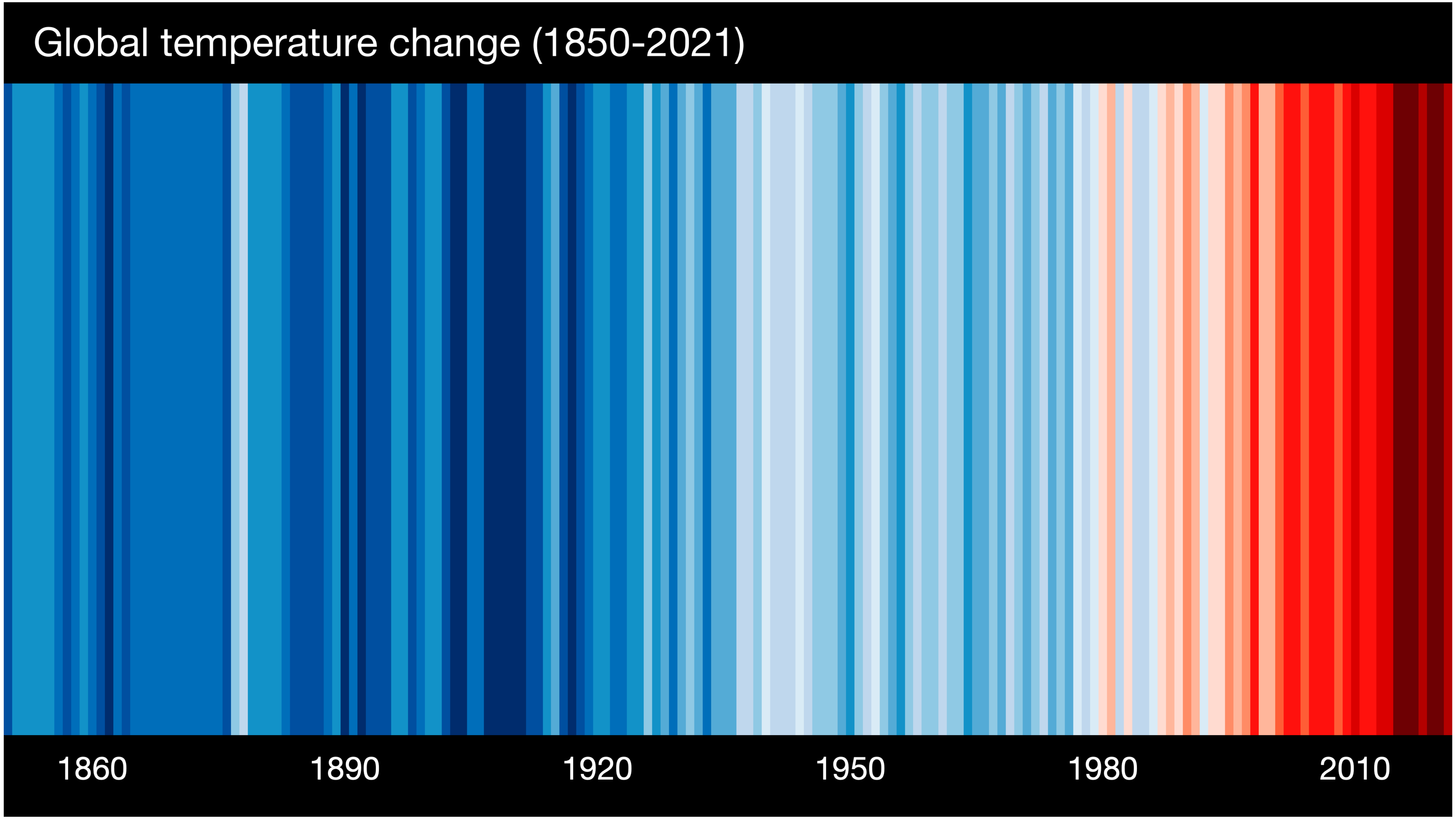
Super high resolution simulation of the ocean showing the dominance of eddies



https://www.youtube.com/watch?v=CCmTY0PKGDs

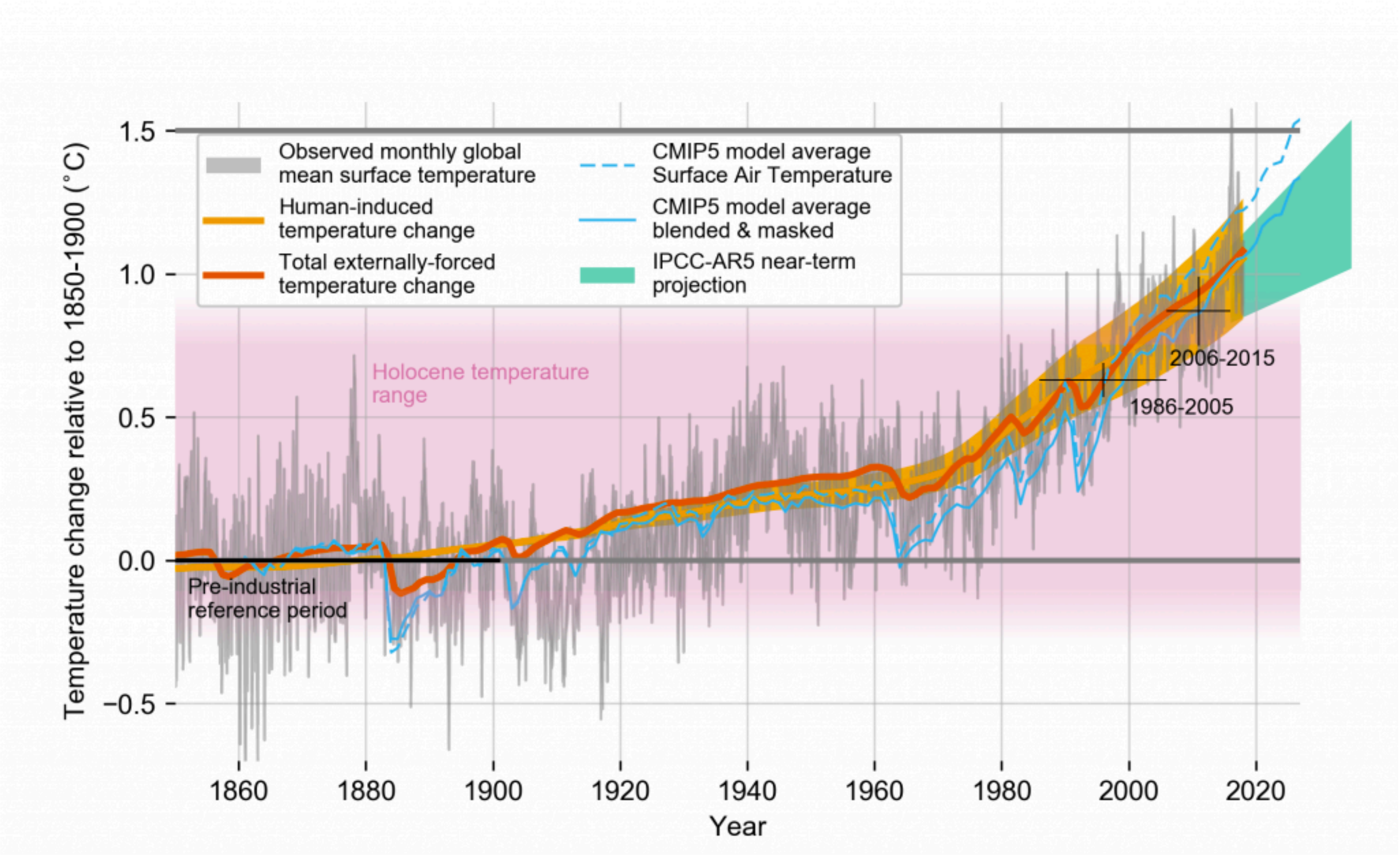
Climate Change: Pictures tell the story

Global Average Temperature: the famous stripes (Met Office)



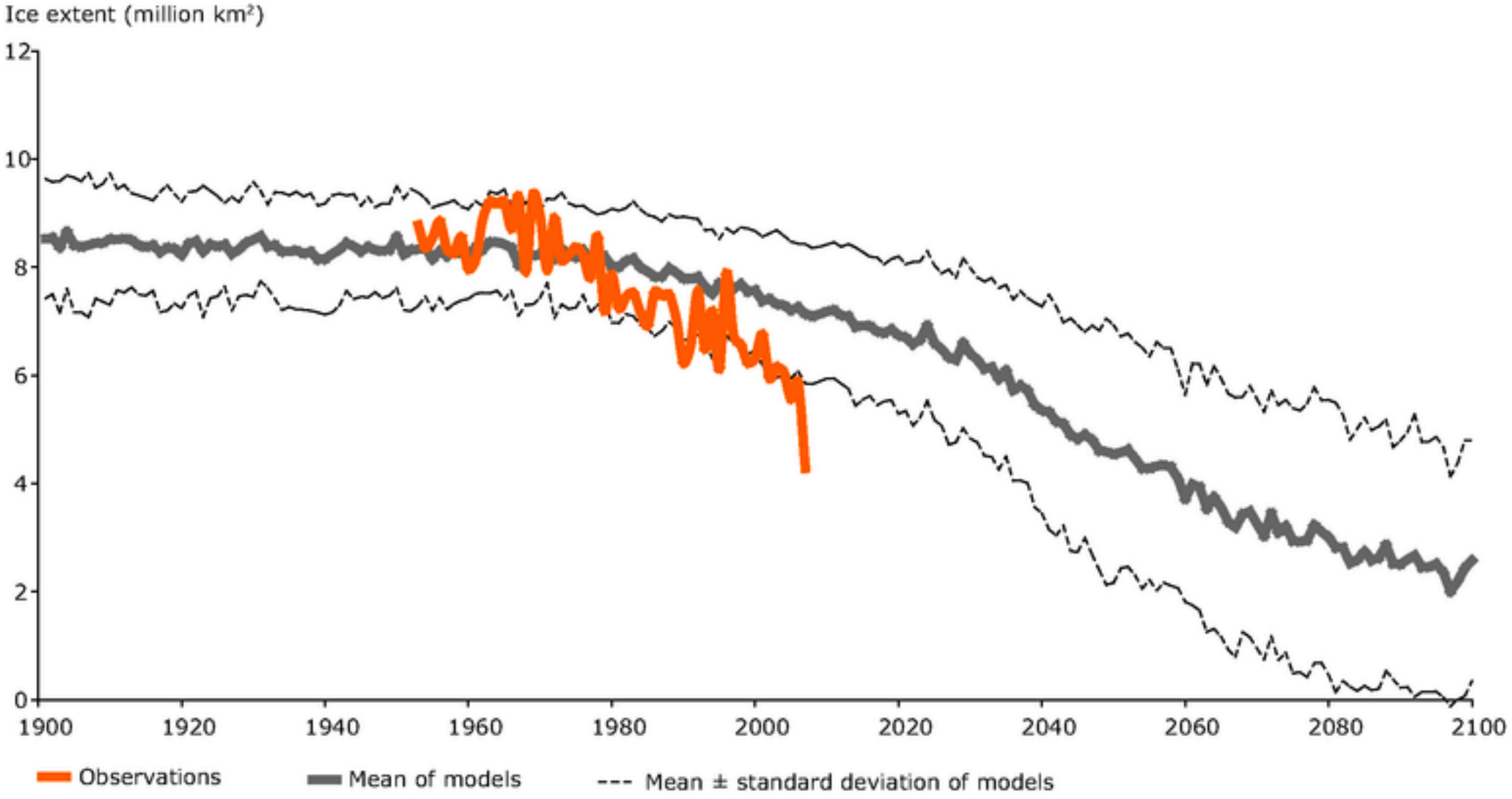
A graph so good it is a design phenomenon: https://chezvoila.com/blog/warmingstripes/

Global Average Temperature (IPCC)



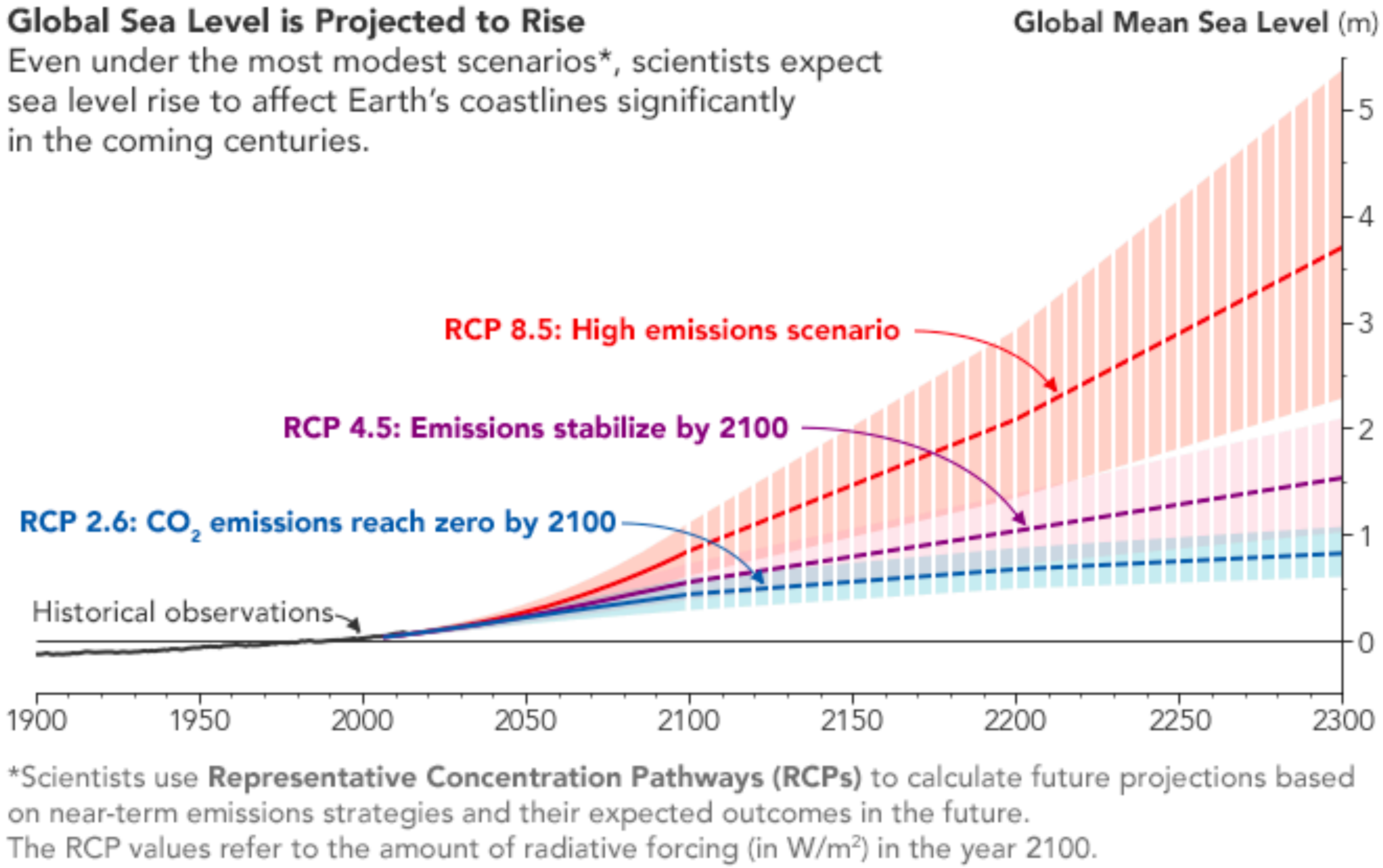
Dates from 2016 or so. Hard to get a sense of what a global average means

Sea Ice Extent (European Union)



Dates from 2008 or so. Observations were already lower than one standard deviation of models.
https://www.youtube.com/watch?v=U-REWv1UhO8

Sea Level Rise Projections (NASA Earth Observatory) Low lying areas are particularly badly affected, check out this reconstruction of Florida. https://www.youtube.com/watch?v=9bplDxpCDII

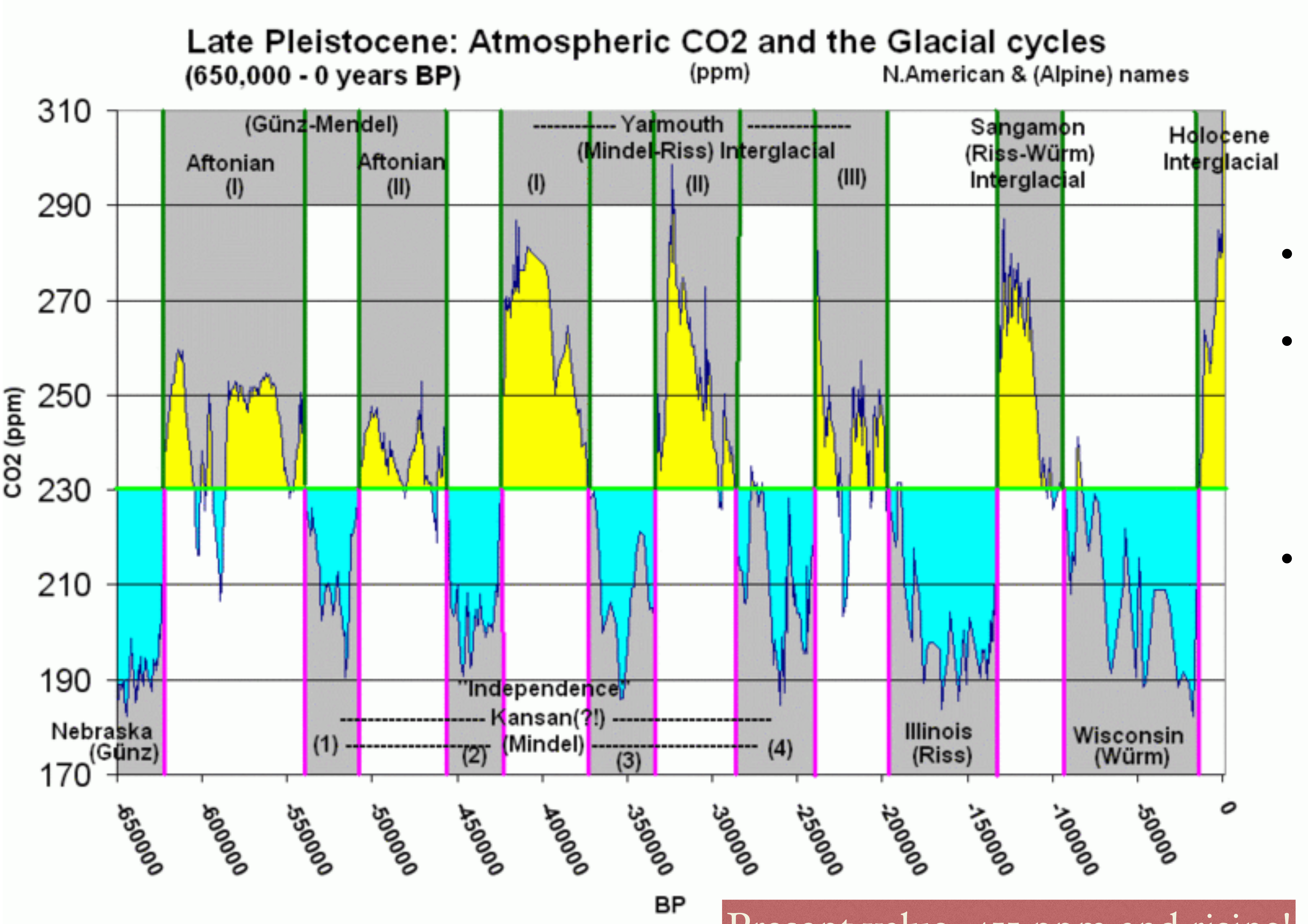


Global Sea Level is Projected to Rise
Even under the most modest scenarios*, scientists expect sea level rise to affect Earth's coastlines significantly in the coming centuries.

*Scientists use **Representative Concentration Pathways (RCPs)** to calculate future projections based on near-term emissions strategies and their expected outcomes in the future.
The RCP values refer to the amount of radiative forcing (in W/m^2) in the year 2100.

Sea level has changed a lot in the past, and the predicted changes by 2100 are not extreme, but they can lead to huge problems because the humans preferentially inhabit regions close to sea level and because sea is dynamic. Hence a storm surge may reach much further inland, into inhabited regions, even with a small amount of permanent sea level change.

Lecture 2: The Past: how we learn about climate

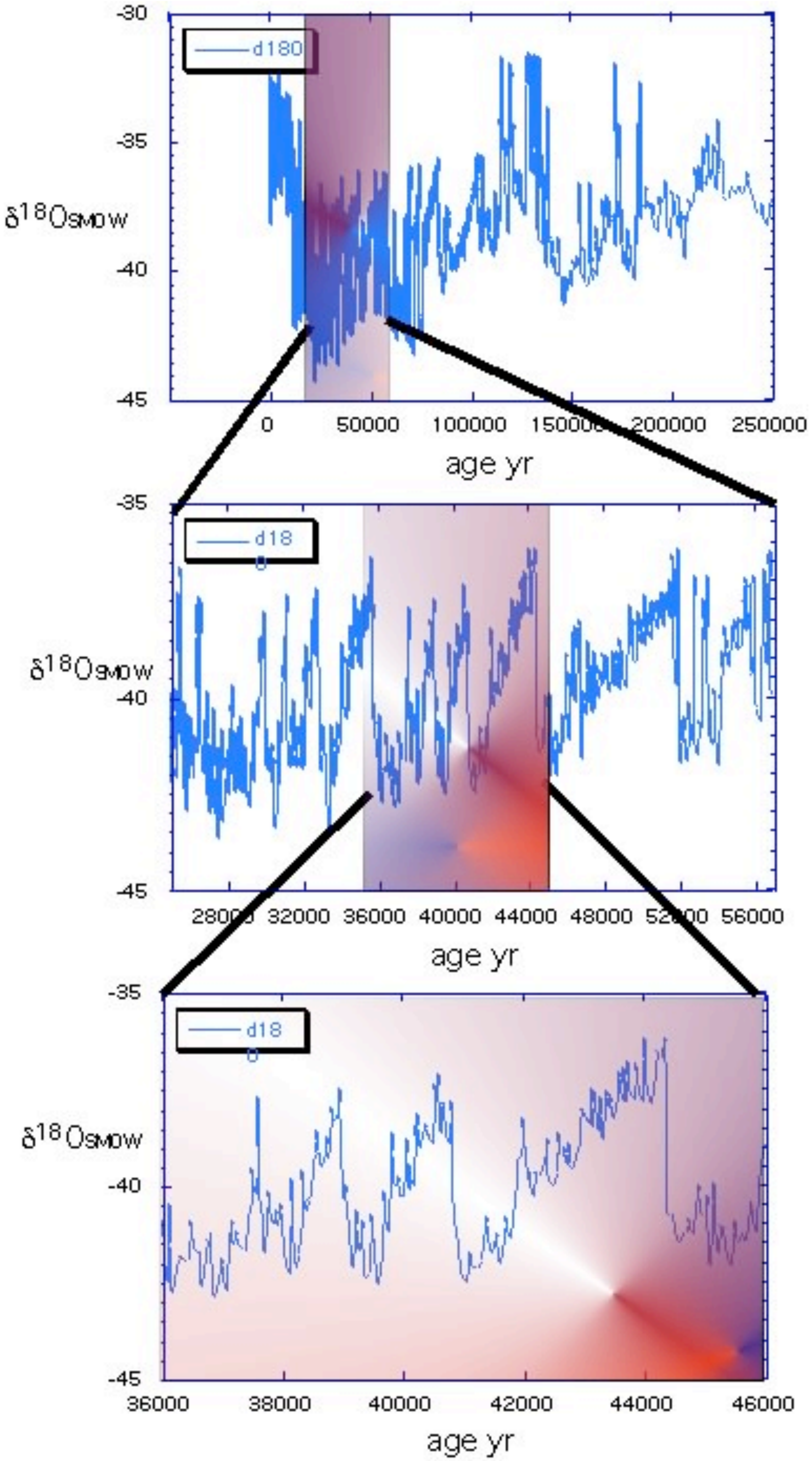

Late Pleistocene: Atmospheric CO2 and the Glacial cycles

(650,000 - 0 years BP)

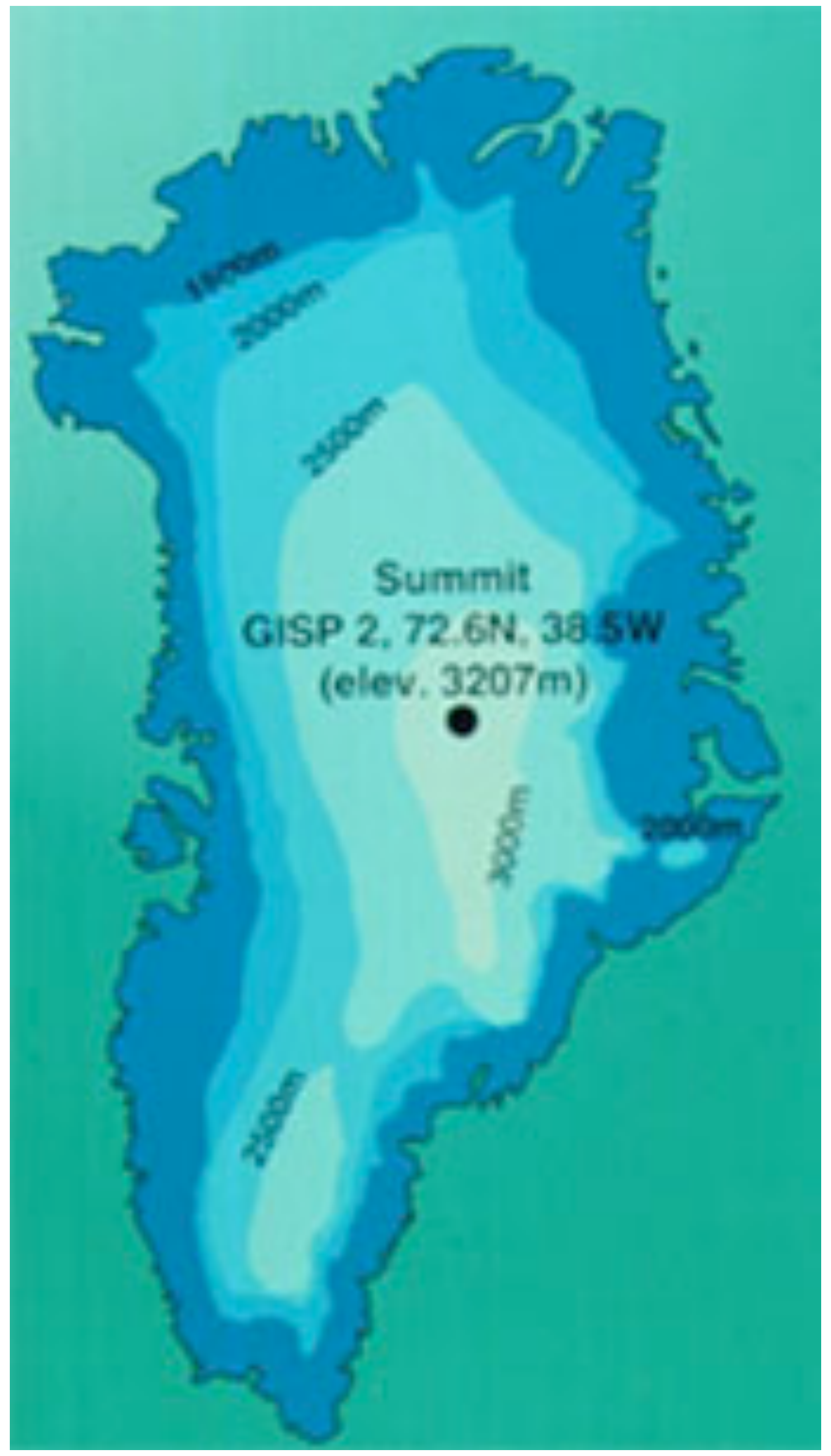
Present value: 411 ppm and rising!

- Climate changes on many timescales.
- We can get evidence about the past from geology, sedimentology and other sciences and get an idea of possible climates.
- With improving technology we can refine the temporal record of climate.

Temperature is reconstructed from ice bubbles in ice cores drilled near the top of Greenland. The deeper you go the older the bubbles.
The ratio between the concentration of the two isotopes of oxygen O^{16} and O^{18} changes with atmospheric temperature and this tells us about past temperatures.



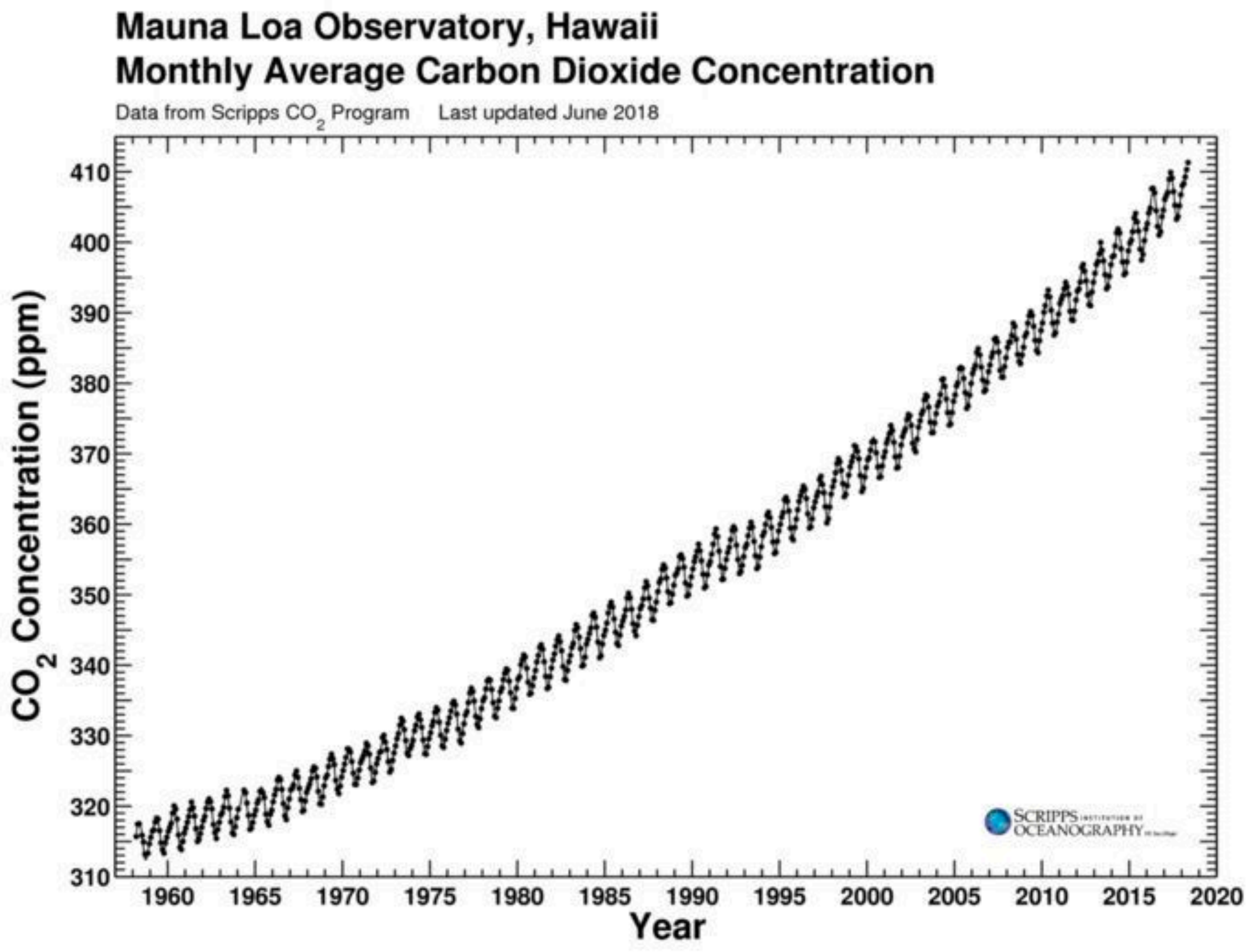
The GRIP and GISP ice records tell us about varitations in the climate during the glacial period and subsequent deglaciation



Systematic variations can be seen at various timescales and each could serve as a research focus.

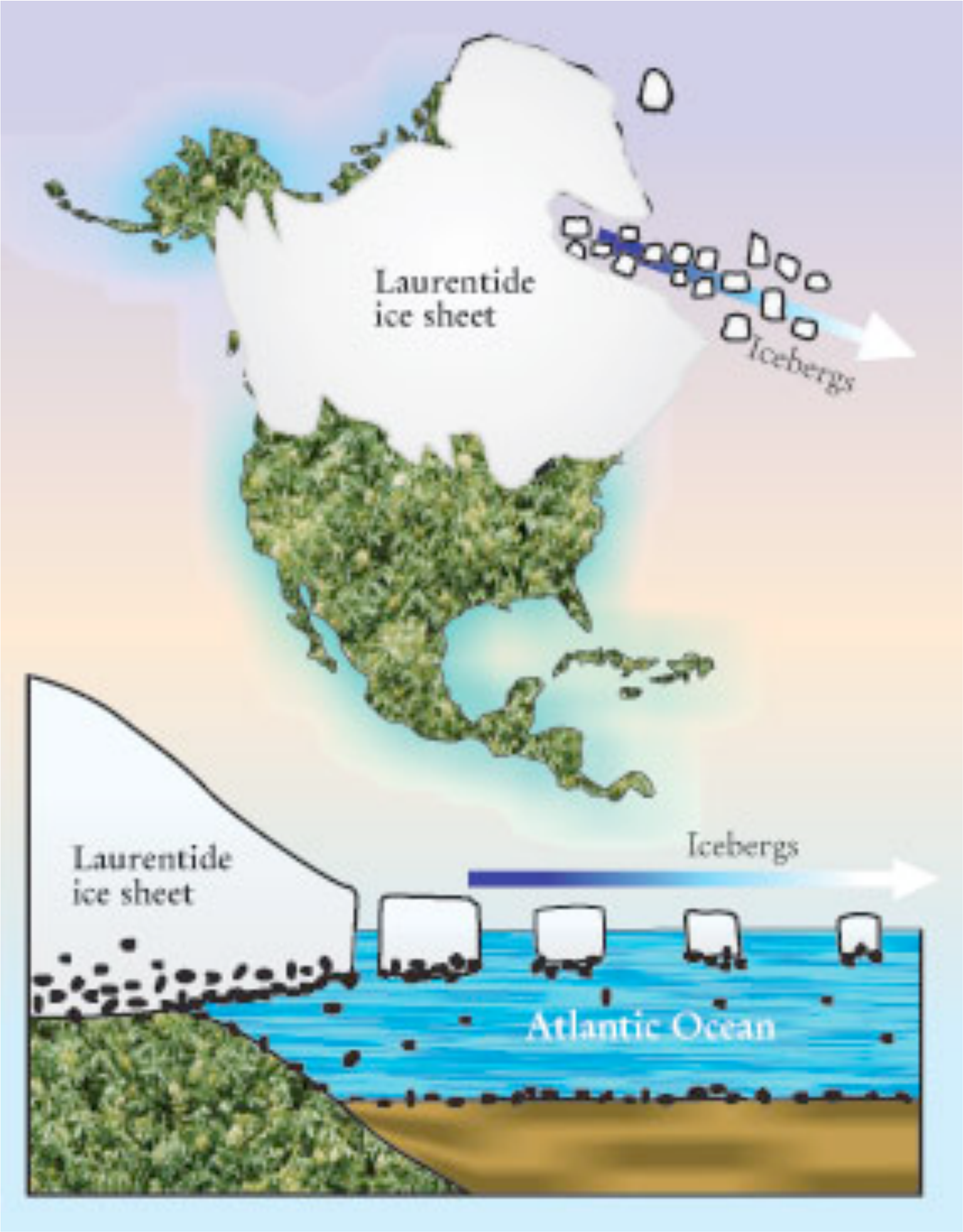
Climate's time series

- To a mathematician the pictures on the last two sides are pretty exciting.
- This is because they are both relatively simple (just a plain x-y graph) but with complex features (e.g. variations on many scales).
- We will have to leave the mathematical details of how one might analyze time series for your future explorations but I can discuss in the Q&A.
- Here we just note some basics: one would like to unambiguously identify trends (e.g. for the Keeling curve on the lower right).
- We would like a way to identify what time scales matter.



Mauna Loa Observatory, Hawaii
Monthly Average Carbon Dioxide Concentration

Data from Scripps CO_2 Program Last updated June 2018

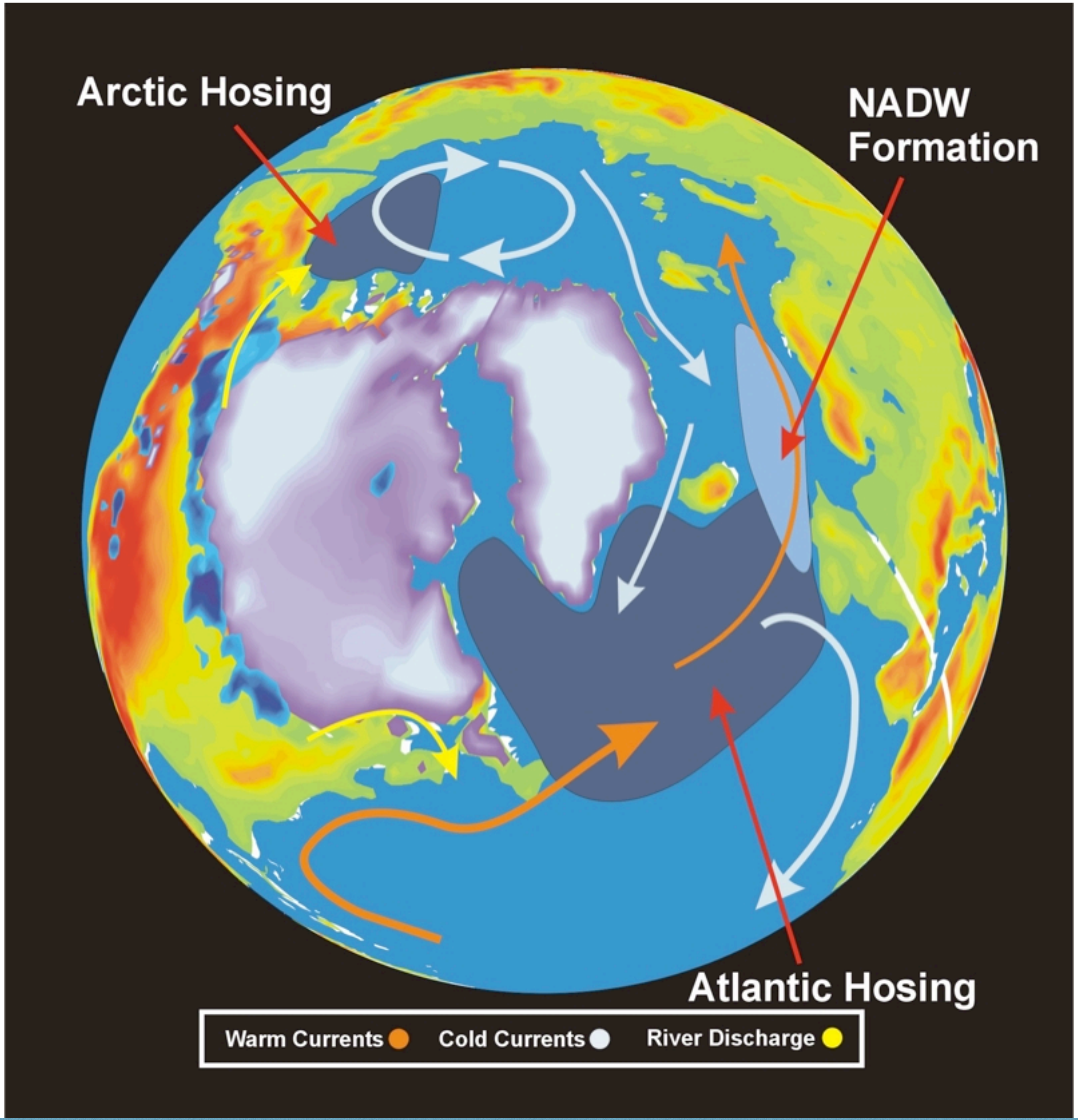


Ice rafted debris with an Arctic origin found in the mid Atlantic suggests episodic ejection of "iceberg armadas" from the Laurentide ice sheet.

These are called "Heinrich Events" and successful geoscience often has this "fancy diagram" aspect to it.

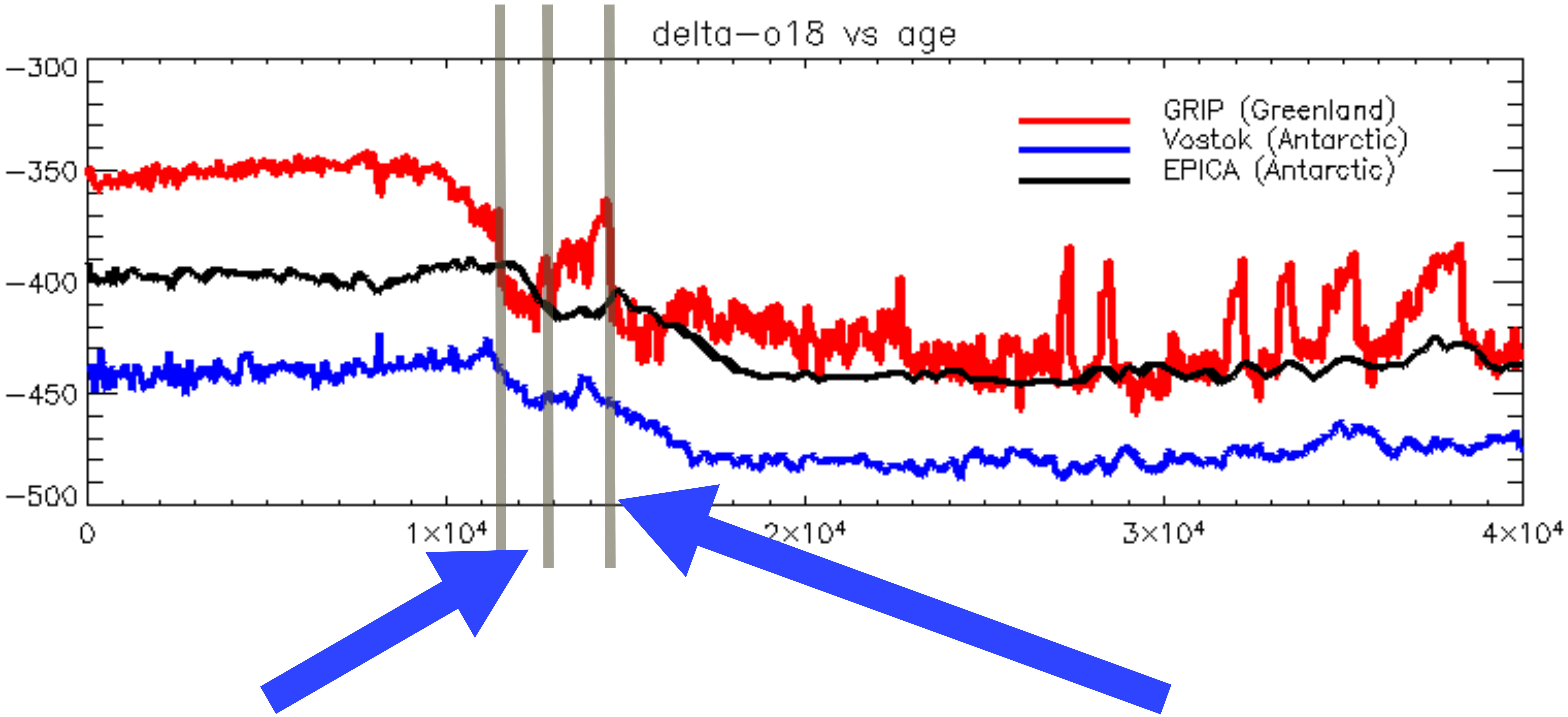
By the standards of physics this is a highly inductive, and even speculative argument, but…

This idea is generally attributed to Syukoro Manabe, who won 1/4 of the 2021 Nobel prize in physics.



https://www.nobelprize.org/prizes/physics/2021/manabe/facts/

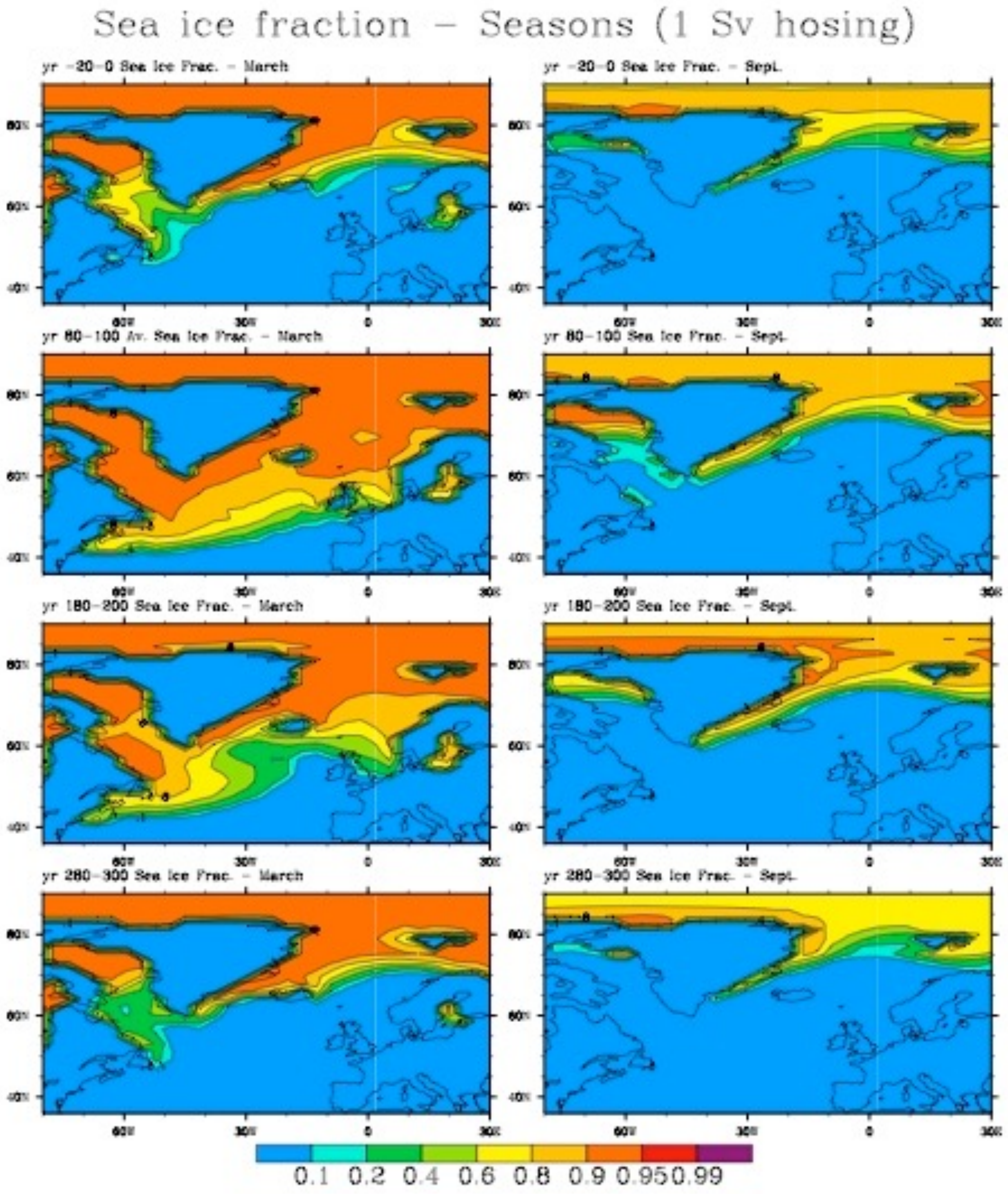
Late Glacial and Deglaciation Chronology



During the millenium or so long Younger Dryas (YD) period temperatures return to near glacial values.
The YD is a blip in the record but was huge event locally!

Meltwater pulse 1A is postulated to end the last glaciation

Climate Models

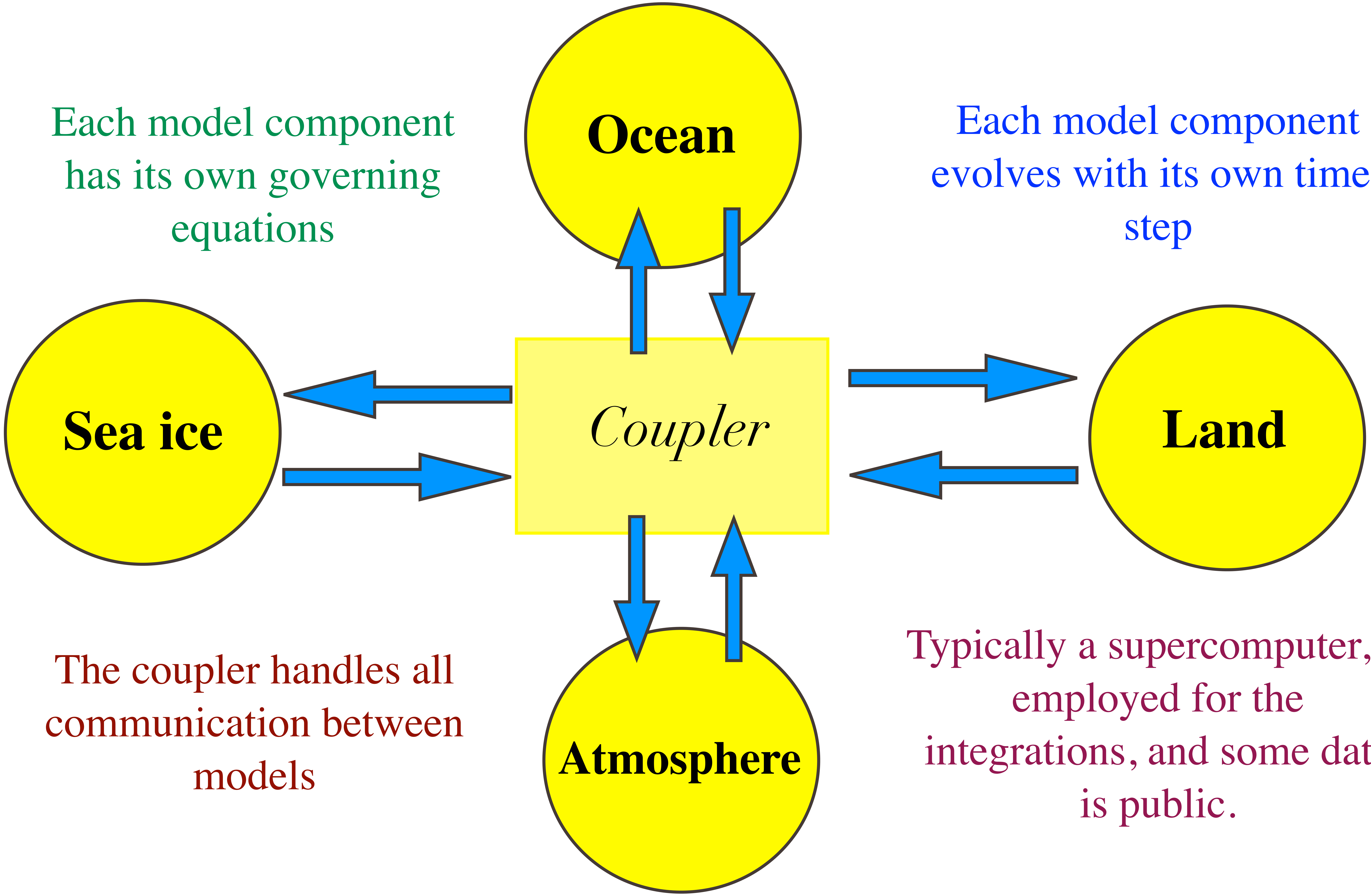


Climate Models

- The climate system follows certain rules, largely based on physics.
- These can be turned into computer code and this code is run to produce simulations of aspects of the climate system.
- Of course what comes out is just a bunch of numbers. How those numbers are organized is an interesting computing problem (see for example the wiki page for NETCDF: https://en.wikipedia.org/wiki/NetCDF).
- When data is well organized it can be used to generate meaningful pictures, like the ice extent picture on the last slide.
- Only when data is converted to “graphs” or “images” can it be interpreted.
- Science needs the math, but the math must be expressible in digestible form.

Climate Modeling: Coupled Model Structure

In the early 2000s the notion of a “coupler” was cutting edge software. Now it is standard, but open data, and open big data in particular, continues to be an active challenge.



Each model component has its own governing equations

Each model component evolves with its own time-step

The coupler handles all communication between models

Typically a supercomputer, is employed for the integrations, and some data is public.

Climate’s computing challenges

- Perhaps the second most important aspect of climate science is that climate models are difficult to learn about and difficult to run.
- Some time ago we passed the point where data analysis is as large of a task as the actual model construction and running.
- But as the social side of computing has evolved (open source, open data, etc) so to has the set of tools we can bring to bear on the climate problem.
- An important “tether” is that some problems consistently pop up no matter how sophisticated the model is.
- Even climate model experts can miss the key assumptions behind a well-accepted model (though it is important to say that this is now way says climate models are wrong).

Climate Basics: Stefan’s Law

Stefan’s Law as a Calculus problem

- Stefan’s law is en expression for the energy radiated by an ideal black body with a given temperature (in degrees Kelvin).

- $E(T) = \sigma T^4$ where σ is a physical constant, see its wiki page for more: https://en.wikipedia.org/wiki/Stefan%E2%80%93Boltzmann_constant#:~:text=The%20Stefan%E2%80%93Boltzmann%20constant%20(also,which%20is%20proportional%20to%20the)

- $E(T) = \sigma T^4$ is a very rapidly increasing function. We can differentiate to get $E'(T) = 4\sigma T^3$ and hence the linearization reads $E_l(T; T_0) = \sigma T_0^4 + 4\sigma T_0^3(T - T_0)$ which we can write as a differential: $\Delta E = 4\sigma T_0^3 \Delta T$

- If we linearize at (0, 20, and 40) degrees centigrade (273, 293, and 313 degrees Kelvin) we get slopes of (4.61,5.70,6.95). In other words the fact that the constant is small means the slopes don’t change too wildly in our comfort zone of temperatures.

- If you instead consider the surface of the Sun (5,800 Kelvin) the slope is a (somewhat crazy) 4.4×10^4.

- The moral of the story is that when it comes to thermal physics, nonlinearity matters, and matters more as temperature rises.

Stefan’s Law: The Basics

- Ultimately planetary climate I about energy.
- Almost all energy for a planet like ours comes from the sun (you can envision interesting sci fi scenarios for other set ups, e.g. a moon of Jupiter).
- The Sun has a few key features: it is very large compared to the Earth, almost unimaginably hot, but to balance both of those it is also very very far.
- We will put some math to that when we talk about the geometry of orbits, orbital parameters and their effect on climate.
- Energy from the Sun consists of electromagnetic radiation, and we will discuss what that means in the next chapter.
- For this introduction we merely state the fundamental law of heat energy as a function of temperature, or Stefan’s Law: $E(T) = \sigma T^4$ where σ is a constant, see its wiki page for more: https://en.wikipedia.org/wiki/Stefan%E2%80%93Boltzmann_constant#:~:text=The%20Stefan%E2%80%93Boltzmann%20constant%20(also,which%20is%20proportional%20to%20the)

Stefan’s Law as Science fiction

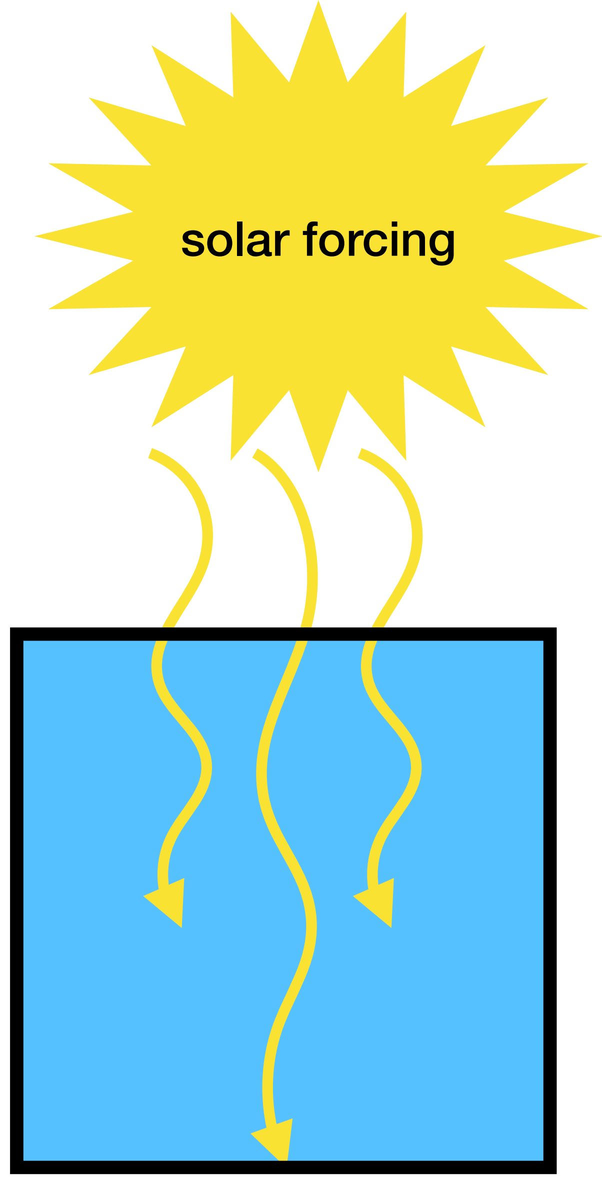
- Now let’s imagine a Universe in which Stefan’s Law had a different power. Let’s even assume it has a pretty similar shape qualitatively, take
- $E_{sci-fi}(T) = \sigma T^2$ where σ is a physical constant.
- We can again differentiate to get $E'_{sci-fi}(T) = 2\sigma T$ and we can write this as a differential: $\Delta E_{sci-fi} = 2\sigma T_0 \Delta T$
- If we linearize at (0, 20, and 40) degrees centigrade (273, 293, and 313 degrees Kelvin), and keep the same value of the constant as in our Universe, we get slopes of $(3.1, 3.3, 3.5) \times 10^{-5}$ respectively.
- This would mean that energy would change very slowly in this Universe!

Lecture 3: Energy Balance and Climate

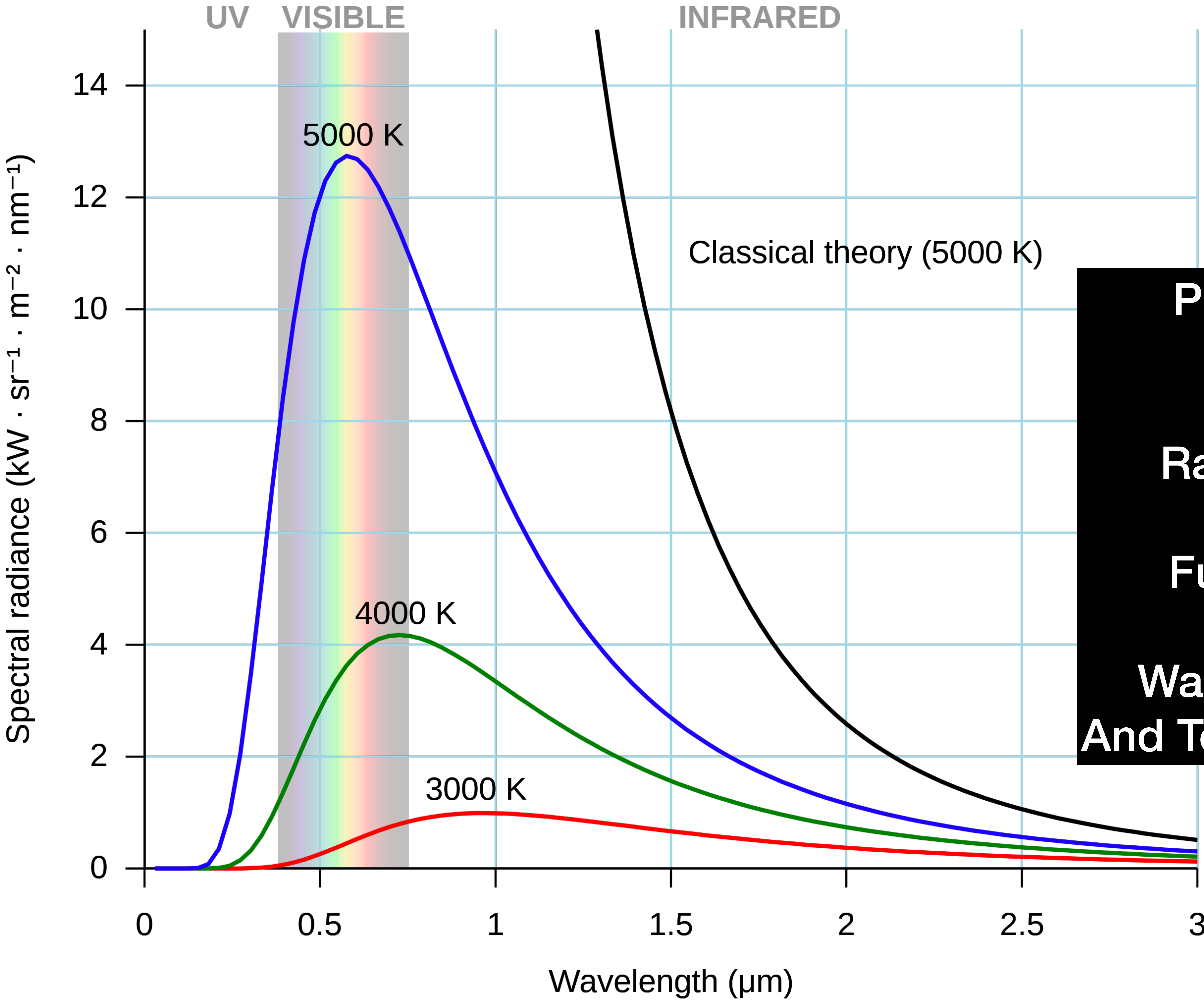
Marek Stastna

- The vast majority of the energy that drives the Earth's climate is from the Sun (the rest is from the core, though we will mostly ignore that portion).
- The Sun is a G type, main sequence star (for more about the Sun info see https://en.wikipedia.org/wiki/Sun)
- For us the main point will be that the Sun is essentially a perfect sphere, and while there is a lot going on in the Sun, the upshot is that the Sun is hot (5,778 Kelvin) and radiates a lot of electromagnetic radiation (63 million Watts per square meter!!!).
- The Earth is approximately 151.71 million kilometres from the Sun, so thankfully all that Energy has time to spread out; sometimes astronomers say planets like Earth lie in the Goldilocks zone.
- If we want to try to understand the energetics of climate (a so-called energy balance model, or EBM) we need to start with the radiation sent out by the Sun, and the portion that is absorbed by the Earth.
- This will neglect all sorts of details that we can subsequently fill in.
- We will start with two "laws", the first of which is Stefan's Law that you have already seen: $E(T) = \sigma T^4$ recall it says energy increases really quickly with temperature.
- The second is Planck's distribution of electromagnetic radiation radiated, and this takes a bit of explaining.

Simplest "EBM"

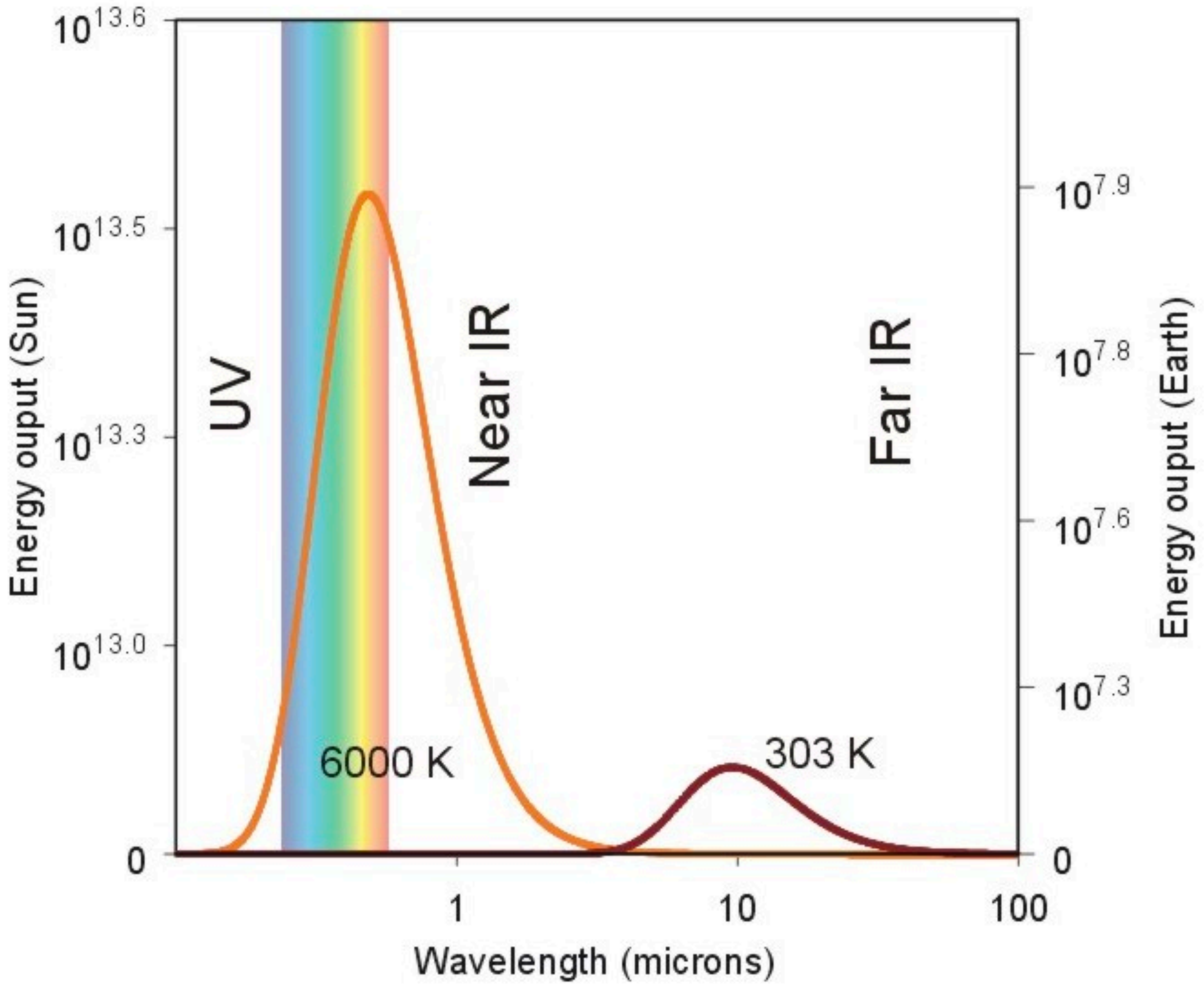


Planck’s Law For Radiation As a Function Of Wavelength And Temperature



- The story of Planck’s law is rather involved and best left to the realm of physics (it is worth noting that it is one of the foundations of why quantum mechanics was developed).
- You can learn more at (this is the source for the previous slide’s graph): https://en.wikipedia.org/wiki/Planck%27s_law#:~:text=Planck's%20law%20describes%20the%20unique,cavity%20with%20rigid%20opaque%20walls
- The upshot is that the distribution of energy is a function with a peak the wavelength of which shifts with temperature.
- The size of the peak also changes and this is in accord with (the simpler) Stefan’s Law.
- From the image on the previous slide you can note that the Sun radiates energy in the visible spectrum (that’s why we can see it, after all).
- A “cooler” object (quotes because 3000 K is still pretty hot) appears to have a rather flat distribution of energy versus wavelength.
- You can look up the functional form of the Planck distribution, but for us the next slide will tell us all we need to know.

- The upshot of Planck's law for climate models is that the Sun sends out most of its energy in the UV to visible range (sometimes the regions are called "bands").
- On the other hand the Earth, whose average temperature is 303 Kelvin sends out energy in the infrared regime.
- The graph is on a log scale and you should note that it's a two axis graph, because what the Earth radiates is so much less.
- The shift in wavelength matters because the Earth has an atmosphere and this atmosphere changes which parts of the electromagnetic wave spectrum can get through and which can't (detail on next slide).



Credit: David Babb, Penn State open Meteo 3 notes https://en.wikipedia.org/wiki/Absorption_(electromagnetic_radiation)#/media/File:Atmospheric_electromagnetic_opacity.svg

Gamma rays, X-rays and ultraviolet light blocked by the upper atmosphere (best observed from space).

Visible light observable from Earth, with some atmospheric distortion.

Most of the infrared spectrum absorbed by atmospheric gases (best observed from space).

Radio waves observable from Earth.

Long-wavelength radio waves blocked.

Nothing gets in

Not much gets out

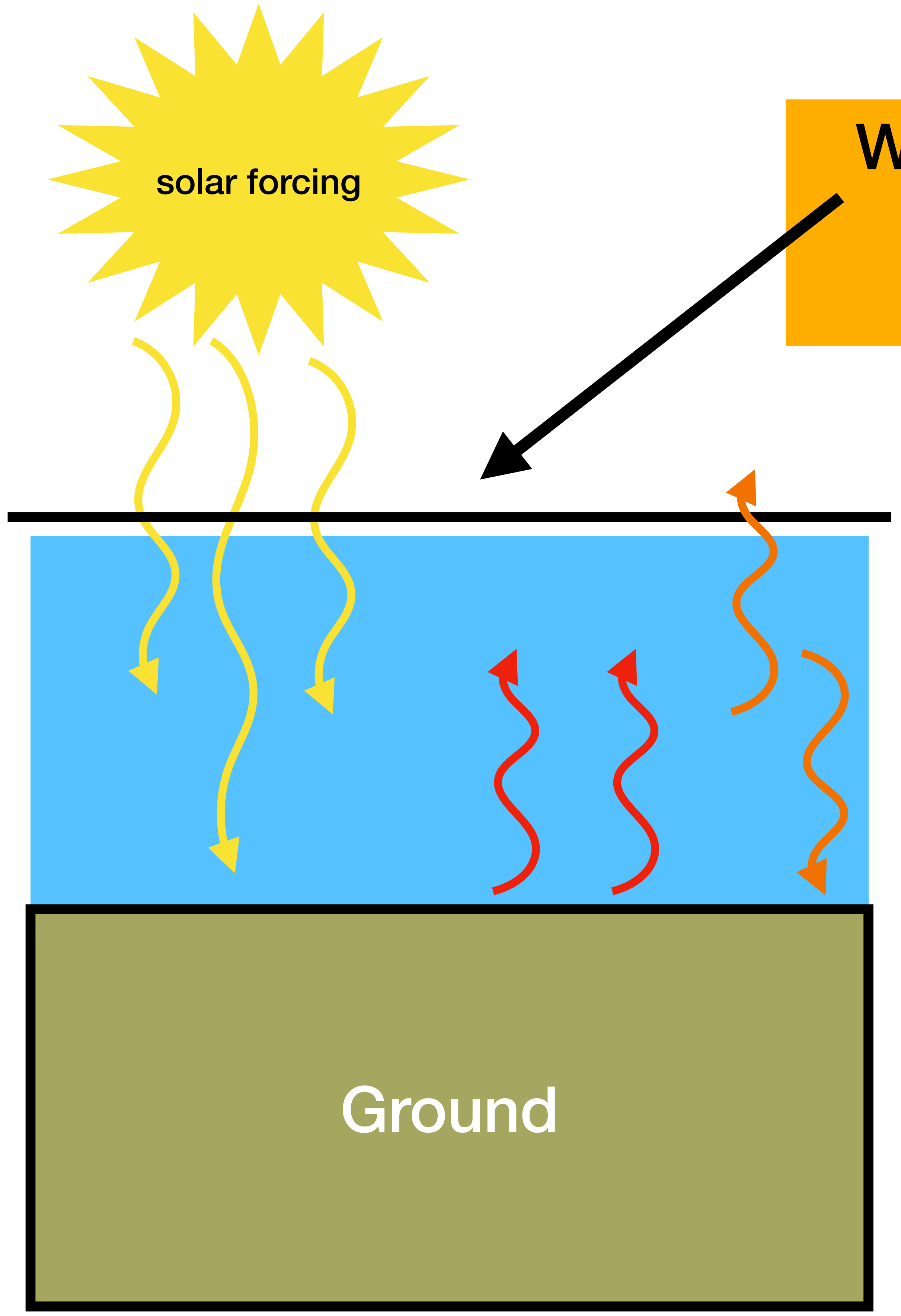
Nothing gets in

Lots gets in

- OK so here is the story so far:
  - We have an energy source from the Sun, which is assumed to be a black body. Call it Q_{sun}
  - We assume the Earth is a black body that absorbs all that energy from the Sun, with a temperature T_{BBE} where BBE stands for Black Body Earth.
  - We assume Stefan's law holds.
- $Q_{Sun} = \sigma T_{BBE}^4$ and we need some numbers.
- We will work out some of the orbital parameter related ones when we discuss the geometry of the Earth-Sun system, but for now take $Q_{sun} = 342.5$ and $\sigma = 5.67 \times 10^{-8}$.
- This gives $T_{BBE} \approx 278.8K$ which is very nearly zero Centigrade.

- There is a big assumption on the last slide, and that is that the Earth successfully absorbs all that incoming energy.
- The variable that determines this is called Albedo and is a fraction between zero and one.
- A black body has an albedo of 0 and a perfect reflector has an albedo of 1.
- Let's denote albedo by a.
- You can see more info on the related wiki page https://en.wikipedia.org/wiki/Albedo
- The modified equation is now: $(1-a)Q_{Sun} = \sigma T_{BBE}^4$ and we need some numbers.
- A common "whole Earth albedo estimate" is $a = 0.3$.
- Let's again take $Q_{sun} = 342.5$ and $\sigma = 5.67 \times 10^{-8}$.
- This gives $T_{BBE} \approx 255K = -18$ Celsius which would imply a frozen, lifeless planet.
- Clearly it is the atmosphere that comes to the rescue, but to build a more complicated model we need to get a conceptual picture first.

EBM with an atmosphere: The Greenhouse Effect



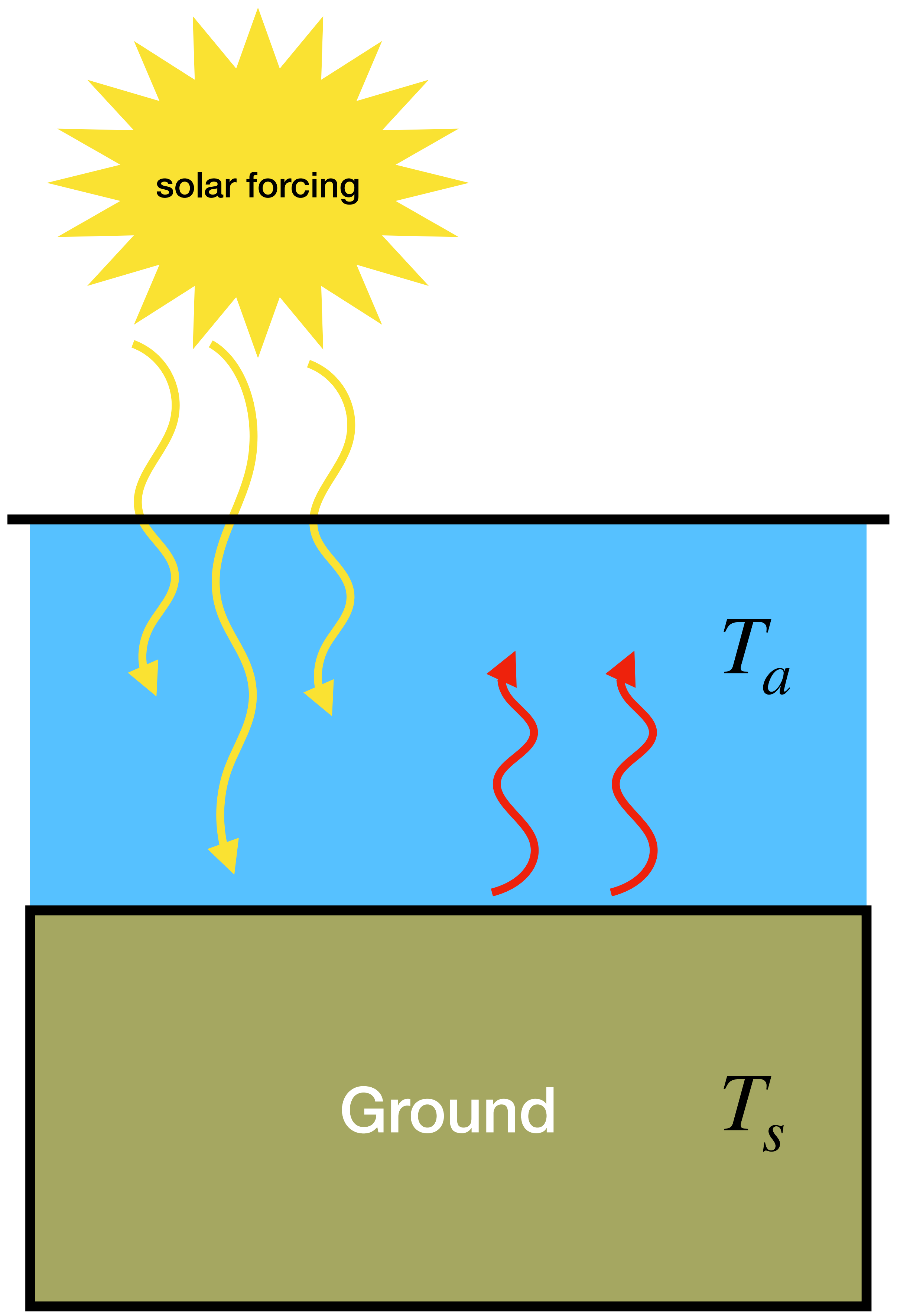
We will write out the energy balance at the top of the atmosphere

Ultraviolet/Visible Radiation in; Passes freely

Infrared Radiation out; Does not pass freely through the atmosphere. The atmosphere also radiates energy

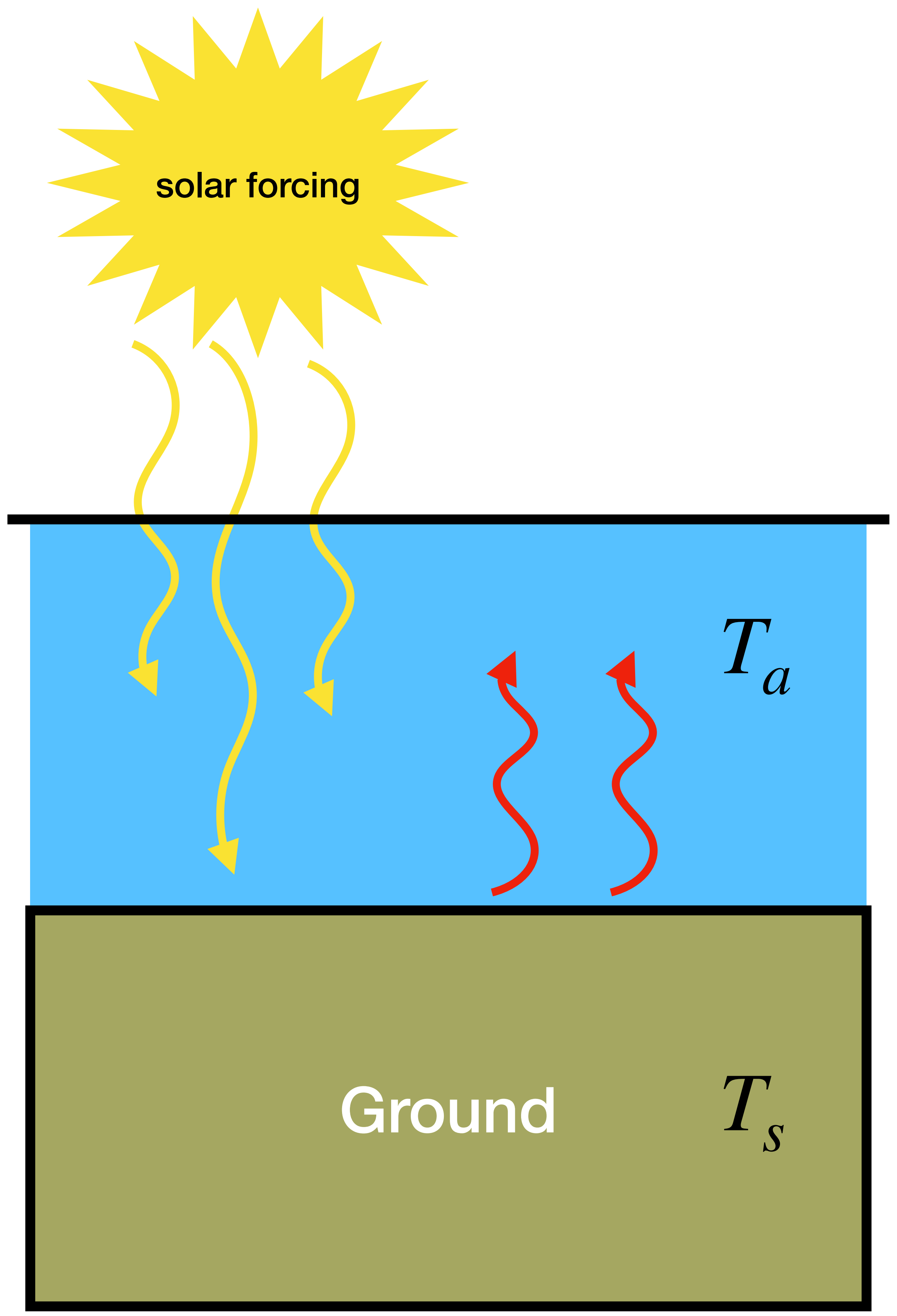
EBM with an atmosphere: The Greenhouse Effect

- Let the atmosphere have a temperature T_a and the ground a temperature T_s.
- a is the albedo of the planet, ϵ measures how strongly the atmosphere absorbs infrared radiation
- **At the top of the atmosphere have:**
- Radiation in from Sun and not reflected=what is radiated by the atmosphere+what is radiated by the ground and manages to get out
- $\frac{S}{4}(1-a) = \epsilon\sigma T_a^4 + (1-\epsilon)\sigma T_s^4$
- **The balance for the atmosphere is:**
- Radiation out to space+radiation down to ground=radiation from the ground
- $2\epsilon\sigma T_a^4 = \epsilon\sigma T_s^4 \implies T_a = 2^{-1/4} T_s$
- **And the equation for the ground is:**
- Energy from space+energy radiated by atmosphere=energy radiated by ground
- $\frac{S}{4}(1-a) + \epsilon\sigma T_a^4 = \sigma T_s^4$



EBM with an atmosphere: The Greenhouse Effect

- Solving gets us an expression for the ground temperature:
- $T_s = 2^{-1/4}\left(\frac{S(1-a)}{\sigma(2-\epsilon)}\right)^{1/4}$
- This is a bit messy, but some sample parameters provide some clarity.
- Fix the solar constant, albedo and Stefan constant: $S = 1370,\ a = 0.3,\ \sigma = 5.670374 \times 10^{-8}$
- Now try a "standard" estimate for the atmospheric absorption, $\epsilon = 0.77$ to get $T_s = 287.95K \approx 15$ Celsius
- A sanity test is to set the absorption constant to zero and this gives $T_s = 255K \approx -18$ Celsius matching our previous result without an atmosphere.
- A super thick atmosphere (maybe like Venus) for which $\epsilon = 1$ gives $T_s = 303.25K \approx 30.25$ Celsius



Simplest “EBM"

- This has been a very satisfying exercise!
- With a very simple set of ideas: black bodies, Stefan’s Law, albedo and an atmosphere that selectively absorbs infrared but not visible radiation we were able to arrive at a model that estimated the temperature of the planet.
- There are still some unknowns (e.g. reconciling this model with what we know about the orbits of planets), but overall an excellent mathematical modelling start.
- What we would like to do next is discuss how to introduce changes in space and time.